Engineering Notes

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Semiempirical Calculations of the Interaction of a Slightly Underexpanded Jet with a Quiescent Atmosphere

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Nomenclature

a = speed of the sound; also $(\gamma - 1)/(\gamma + 1)$, Eq. (10)
c_p = specific heat at constant pressure
h = specific static enthalpy
M^* = Mach number based on the critical speed of the sound
n,p = p_e/p_0 and pressure, respectively
p_t' = stagnation pressure behind a normal shock
r_b,r_e = local jet boundary and nozzle exit radii, respectively
T = temperature
v = velocity in the x direction
X,Y = axial and radial coordinates, respectively
$x,y,\bar{y}$ = X/r_e, Y/r_e, and Y/r_b, respectively
β = free mixing length coefficient
γ = specific heat ratio
μ,ρ = molecular weight and density, respectively

Subscripts

b,c,e = jet boundary, centerline, and nozzle exit, respectively
x = at specific location in x direction
$0,t$ = ambient condition and stagnation condition, respectively
1 = condition at start of isobaric region

A SEMIEMPIRICAL approach developed by Abramovich[1] for determining the flowfield properties of a slightly off-optimum supersonic jet exhausting into a quiescent atmosphere is extended to rocket exhausts typical of Martian retro-conditions. If the soft landing of scientific instruments on Mars is to be accomplished by the use of terminal decelerating rockets, it will be necessary to estimate the extent of descent corridor significantly affected by the jet as well as the extent of the surface interaction. The former problem requires the estimate of the growth of the jet while the latter requires more detail as to the internal structure of the jet. The method described herein allows engineering estimates of both the size and structure of the jet but does not attempt to describe the jet-surface interaction.

The analysis of a high-speed, compressible jet into a near quiescent ambient flow has been carried out by several investigators in the past. For highly underexpanded jets the size of the mixing region is considered negligible with respect to the radius of the inviscid core of the jet, and the rate of growth of the plume is considered to be much greater than the rate of growth of the mixing region. Consequently, the jet expansion is an inviscid problem (e.g., Vasiliu[2]). However, in the case of slightly underexpanded jets, which are typical of Mars landing retrorockets in that the atmospheric pressure is significant, the mixing region grows rapidly with respect to the radius of the jet and in fact encompasses the entire jet a few exit radii from the exit. Analytic techniques have been developed (e.g., Libby[3] and Bloom and Steiger[4]) to determine the flowfield prior to total jet mixing for isobaric jets, but little has been accomplished for nonisobaric jet or jets beyond total mixing.

Abramovich[1] has applied semiempirical results for the turbulent mixing of isobaric jets to the case of real jets in order to obtain engineering estimates of the growth parameters. The basic underlying assumption in this analysis is that an essentially isobaric, turbulent, supersonic flow is established at a distance close to the nozzle exit. This has been supported by experimental measurements and by theoretical computations for initially isobaric exhaust jets. Thus, analysis of the initial and transition regions of the jet prior to the beginning of the isobaric turbulent region can be avoided in so far as they occupy a small amount of the total flowfield. The actual extent of these regions is obtained by considering the unmatched jet as a small perturbation of the matched jet.[1]

The technique is developed for the simple case of unheated jets expanding into a quiescent atmosphere of like chemical composition. The results of this computation are essentially the static temperature, stagnation temperature, velocity, static pressure, and normal shock stagnation pressure at all positions within the exhaust field. In addition, the location of the jet exhaust boundary and sonic line are computed as a function of distance from the exit plane.

To determine the flow conditions at the inception of the turbulent isobaric expansion region, consider the region between stations e and 1. If one assumes that the mass, energy, and momentum fluxes through the lateral boundaries are negligible compared with those at the nozzle, the continuity and energy equations can be written $\rho_e v_e y_{b_e}^2 = \rho_1 v_1 y_{b_1}^2$, and $h_e + v_e^2/2 = h_1 + v_1^2/2$, and the momentum equation is

$$\rho_1 v_1^2 y_{b_1}^2 - \rho_e v_e^2 y_{b_e}^2 = (\rho_e - \rho_0) y_{b_e}^2 \qquad (1)$$

Combining these three equations, one obtains in terms of the mean Mach numbers, M_e^*, M_1^*,

$$M_1^* + \frac{1}{M_1^*} = M_e^* + \frac{1}{M_e^*} + \left[1 - \left(\frac{y_{b_e}}{y_{b_1}}\right)^2\right] \times \left(\frac{\gamma+1}{\gamma-1} - M_1^{*2}\right)\frac{\gamma-1}{\gamma+1}\frac{1}{M_1^*} \qquad (2)$$

Since the left-hand side of Eq. (1) $\simeq 0.0$ for $p_e \approx p_1$, and if one assumes that the variation in the mean axial momentum is due to the resultant of the pressure forces on the lateral boundaries, one has as a rough estimate that $n = p_e/p_0 \approx y_{b_1}^2/y_{b_e}^2$. This equation will be used to compute y_{b_1} in future calculations, and, here, for $n \approx 1$, $y_{b_e} \approx y_{b_1}$. Therefore, from Eq. (2) one has that $M_e^* \approx M_1^*$ or $M_e^* \approx 1/M_1^*$. Experimentally,[1] it is shown that the case $M_e^* \approx M_1^*$ is the correct one for the present conditions and, thus, $M_1 > 1$.

Since x_1 depends upon the characteristics of the flow in the initial and transition regions, the problem of determining x_1

Received September 2, 1969; revision received March 9, 1970. The work reported herein was supported by the Jet Propulsion Laboratory under Contract 92080

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is extremely difficult and remains unsolved. However, one can suppose that the extent of the transition region is very small compared with that of the initial region and assign to the isobaric section the same axial coordinate as the one that corresponds to the transition section. Since an analytic solution for the initial region of an unmatched jet does not exist, the relationship between M_e, p_e/p_0, and x are approximated by the results of the matched jet, i.e., the shear layer as computed for a matched jet having the same exit Mach number and ambient pressure is superimposed on the inviscid solution for the unmatched jet to determine x_1, the intersection of the shear layer with the nozzle axis.

Solutions are given in Ref. 1. For $\gamma_e = 1.25$ and $M_e = 3.98$, one finds that $x_1 \approx 12$ for $n^2 = 1.14$ ($y_{b_1}/y_{b_e} \simeq 1.07$). Thus, the slightly underexpanded jet looks much more like a bounded cylinder than the typical conical pattern approximating the highly underexpanded jet, and the initial region (since $x_1 \approx 12$) represents a small portion of the flowfield when one is concerned with flows out to $x \gg 100$. For this analysis, dT_c/dx must also be small compared to dT_{t_c}/dx so that the heat conduction within the jet is negligible.

Again referring to a control volume extending from the nozzle exit plane to the jet isobaric region, one can write the momentum equation as follows:

$$(v_{c_x}/v_{c_e})^2 A_x = (y_{b_e}^2/y_{b_x}^2) B_e \tag{3}$$

where

$$A_x \equiv \int_0^1 \left[\frac{\rho}{\rho_{c_e}}\left(\frac{v}{v_c}\right)^2 \bar{y}d\bar{y}\right]_x, \quad B_e \equiv \int_0^1 \left[\frac{\rho}{\rho_{c_e}}\left(\frac{v}{v_c}\right)^2 \bar{y}d\bar{y}\right]_e$$

Since the heat conduction has been assumed negligible, the energy equation reverts to the conservation of h_t, i.e.,

$$(v_{c_x}/v_{c_e})\overline{\Delta T}_{t_c} C_x = (y_{b_e}^2/y_{b_x}^2) D_e \tag{4}$$

where

$$C_x \equiv \int_0^1 \left(\frac{\rho}{\rho c_e}\frac{c_p}{c_{p_{c_e}}}\frac{v}{v_c}\frac{\Delta T_t}{\Delta T_{t_c}}\bar{y}d\bar{y}\right)_x, \qquad \Delta T_t \equiv T_t - T_0$$

$$\Delta T_{t_c} \equiv T_{t_c} - T_0$$

$$D_e \equiv \int_0^1 \left(\frac{\rho}{\rho_c}\frac{c_p}{c_{p_c}}\frac{v}{v_c}\frac{\Delta T}{\Delta T_{t_c}}\bar{y}d\bar{y}\right)_e, \qquad \overline{\Delta T}_{t_c} \equiv \frac{\Delta T_{t_c}}{\Delta T_{t_{c_e}}}$$

In order to compute the size of the jet at a given position, one might refer to the equation of the jet for an incompressible fluid, which is given by $dy_b/dx \simeq v_c/v_{ch}$, where v_{ch} is a characteristic velocity of the mixing process. Yakolevskii[5] has proposed that the same formula can be used to describe the growth for turbulent, compressible supersonic jet mixing if one has chosen the proper v_{ch}. Several v_{ch}'s have been proposed, but the following form is chosen in that it gives acceptable accuracy in comparison with experiment[1,5] and yields a relatively simple result to compute:

$$v_{ch} = \rho_c v_c/(\rho_0 + \rho_c) \tag{5}$$

Therefore, one obtains

$$dy_b/dx = (1 + \rho_0/\rho_c)\beta/2 \tag{6}$$

where β is a coefficient related to the free mixing length and has been found experimentally Eq. (5) to be ≈ 0.22 for cases similar to the slightly off-optimum jets we are considering. However, since the jet is isobaric, for $\mu_c = \mu_0$ and high velocity jets, Eq. (6) takes the form

$$\beta dx/dy_b = [1 - (M_e{}^{*2}/2)(v_c/v_{c_e})^2]^{-1} \tag{7}$$

Equation (3) in the preceding discussion was for an isobaric jet. If one considers a slightly off-designed jet, the momentum equation becomes,

$$2\int_0^{y_b} \rho v^2 y dy = [p_e v_e^2 - (p_e - p_0)]y_e^2 \tag{8}$$

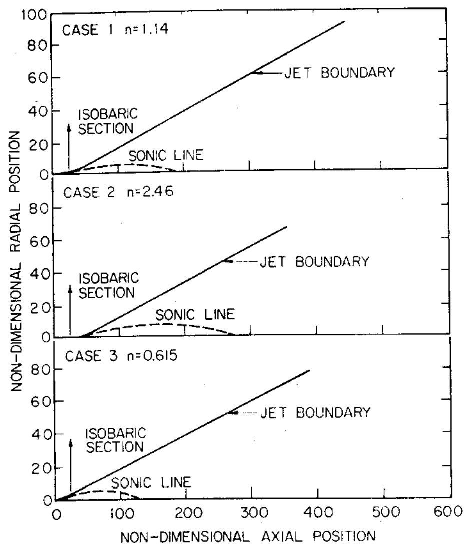

Fig. 1 Extent of jet expansion into a quiescent atmosphere. (Linear dimensions nondimensionalised by nozzle exit radius.)

and after several transformations, Eq. (8) can be written as

$$v_c^2 A' = N/y_b^2 \tag{9}$$

where

$$A' = \int_0^1 \frac{2(1 - aM_e{}^{*2})(v/v_c)^2}{1 - aM_e{}^{*2}(v/v_c)^2} y_b dy_b; \; a \equiv \frac{(\gamma - 1)}{(\gamma + 1)}$$

$$N = B_e n + (1 - aM_e{}^{*2})/(aM_e{}^{*2})(n - 1)a/(1 + a)$$

and B_e is given by Eq. (3). The simultaneous solution of Eqs. (4, 7, and 9) yields the following:

$$\beta(x - x_1) = \left[\frac{N}{0.134(1 - aM_e{}^{*2})}\right]^{1/2}\left[\frac{F(z)}{\bar{v}_c} - F(z_e)\right] \tag{10}$$

$$z = (aM_e{}^{*2}\bar{v}_c/2)^{1/2}, \quad z_e = (aM_e{}^{*2}/2)^{1/2}, \quad \bar{v}_c = v_c/v_{c_e}$$

$$F(z) = \frac{1}{(1 - z^2)\xi} - \frac{0.528z^2}{1 - z^2}\xi - 1.07z \ln\left[\frac{\xi + 1.378z}{(1 - z^2)^{1/2}}\right]$$

and $\xi \equiv (1 + 0.0896z^2)^{1/2}$. Note that Eq. (10) gives $\bar{v}_c$ as a function of x and other known quantities.

To find $T_{t_c}(x)$, divide Eq. (3) by (4) to get

$$\overline{\Delta T}_{t_c} = \bar{v}_c(A_x D_e/B_e C_x) \tag{11}$$

where B_e and D_e are known from the potential solution of the internal flow in the nozzle; however, A_x and C_x depend on the

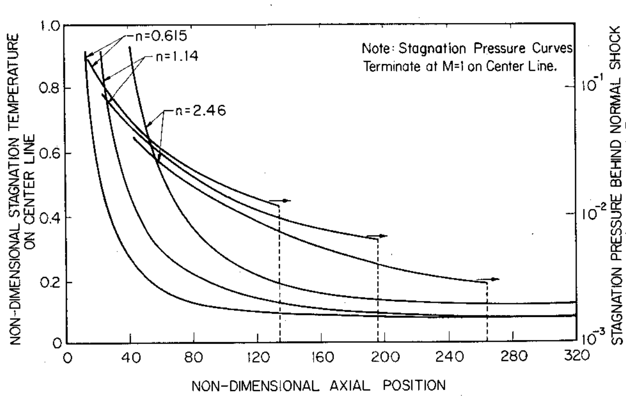

Fig. 2 Axial stagnation pressure behind normal shock and stagnation temperature distributions along center line. (Linear dimension nondimensionalised by nozzle exit radius; pressure nondimensionalised by chamber pressure temperature nondimensionalised by chamber stagnation temperature.)

Table 1 Data input summary ($\mu_e = 21$; $M_e = 3.98$; $T_{t_e} = 5670°R$; $\gamma = 1.25$; $r_e = 0.398$ ft)

Case	p_0, mbar	T_0, °R	μ_0	P_e, psia	p_e/p_0	x_1/r_e
1	20	360	32	0.330	1.14	24.4
2	5	495	31	0.179	2.46	42.6
3	20	360	32	0.179	0.613	14.9

profiles at station x. For the v and T_t profile we use[1]

$$(v/v_c)_x = [1 + (y/y_b)^2]_x^{3/2} \tag{12}$$

$$(\Delta T_t/\Delta T_{t_c})_x = 1 - (y/y_b)_x^{3/2} \tag{13}$$

This procedure is limited to an isobaric jet, since Eq. (11) [deduced from Eqs. (3) and (4)] is based on this assumption. The effect of nonuniform pressure is introduced in Eq. (11) when one substitutes for $\bar{v}_c$ the values obtained for the slightly underexpanded jet given by Eq. (10).

If the physical growth of the plume is desired, Eq. (7) can now be integrated to yield $y_b(z)$. Furthermore, Eqs. (12) and (13) provide sufficient information to determine all the flow-field variables throughout the jet.

A numerical calculation along the foregoing lines is feasible, but if, following the philosophy of this exposition, one is interested primarily in a simple method for reasonable estimates of the plume interaction with the atmosphere, one can take advantage of the fact that A_xD_e/B_eC_x does not vary significantly for different choices of profiles and can be taken as 0.745. The following calculations have been done under this assumption; thus, one does not need to know the internal flow in the nozzle, but only the average values of M_e, p_e, and T_{t_e}.

The preceding analysis was programed in Fortran IV for the IBM 360 computer. Three cases were computed as typical of Mars landing requirements; the input parameters are listed in Table 1. For each case the following constraints were imposed: the descent to Mars was along a local vertical to a smooth plane surface. There was no atmospheric wind present, and the exhaust gases were invariant in composition.

The results of these computations are summarized in Figs. 1 and 2. From these results one may obtain the desired boundaries for which a significant effect of the exhaust jet is experienced, e.g., $p_{t_c}' = 1.1\ p_0$, and $T_t = T_0 + 10°R$. These effects are not significant within the range of axial locations shown in the figures; however, the locations for this T_t effect are 3200, 3900, and 2300 exit radii, respectively. In each case the significant rise in surface pressure will occur before the rise in temperature.

The assumptions involved in these results should be reiterated. In using Eq. (6) for the propagation of a jet, an incompressible law has been extended to a supersonic flow. The validity of this extension has been confirmed by experiments[5] for specific types of jets. Also in this law, the parameter β has been established for flows similar to those considered herein to be 0.22, and the function of v_{ch} used in this equation is a relatively simple expression which has been used to facilitate the mathematics. Better approximations for v_{ch} with the attendant complications to the mathematics are known to exist but were not used in this approximate analysis. Finally, in this equation for the growth of the jet, it was assumed that $\mu_e = \mu_0$. To extend the technique to nonequal molecular weights, another equation would be required. (However, $\mu_e/\mu_0 = 21/31 = 0.7$ should not have a strong effect.)

The profiles of velocity and stagnation temperature assumed in this writing have been confirmed by experiments.[1,5] It is important to note that these profiles will affect most significantly the distribution of Mach numbers. Since the distributions of other variables enter through integrals in the determination of A_x, B_e, C_x, and D_e, these coefficients are not very sensitive to the exact profiles.

Although $p_{t_c}'(x)$ is computed from normal shock relations and thus is accurate, the transport of energy due to turbulence is not considered in this calculation. However, it is felt that the turbulence transport is sufficiently small at large downstream locations that the inviscid solutions to the shock problem are adequate.

References

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Hele-Shaw and Porous Medium Flow for Space Fuel Cells

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THIS Note presents an investigation of two types of flow which may eliminate flow problems in certain space missions. The two flow systems which have been investigated are very similar, and consist of fuel tanks filled with porous, medium, or Hele-Shaw cells. (The Hele-Shaw cells are basically thin parallel membranes or plates placed close together in a fuel tank.) The main changes these devices produce are as follows: 1) the inertia forces are reduced due to the increased viscous action of the surfaces; 2) the Bond number takes on a value less than one, and sloshing is eliminated as a problem; 3) surface tension forces act over a small local region, although they increase; 4) funnelling can be more readily controlled by adjusting the porous medium permeability or Hele-Shaw plate spacing at the fuel tank outlet; and 5) an interface instability may develop at the interface of the liquid fuel and driver gas or liquid.

The experimental results that were obtained came from the use of a Hele-Shaw channel, and the porous medium behavior was inferred from an analogy which exists between the two types of flow. This analogy can be shown quite readily by comparing the functional form of the equation for the mean velocity across a Hele-Shaw channel with the average velocity in a porous medium. The average velocity, **u**, across a Hele-Shaw channel is given by[1]

$$\mathbf{u} = -b^2\nabla P/(12\mu) \tag{1}$$

where b is the distance between the parallel plates, μ the fluid viscosity, ∇P the gradient of pressure, and **u** is the velocity vector. For flow in a porous medium the average velocity is given by Darcey's law as

$$\mathbf{u} = k\nabla P/\mu \tag{2}$$

Presented as Paper 69-678 at the AIAA Fluid and Plasma Dynamics Conference, San Francisco, Calif., June 16–18, 1969; submitted June 11, 1969; revision received October 20, 1969.

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